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Critical probabilities in percolation on decorated graphs

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Abstract. We consider bond percolation on a graph Γ and on a decorated graph Γ^G derived from Γ by replacing every edge in Γ by a finite connected two-rooted graph G. We show that the critical exponent β is invariant under such decorations and that, by varying G, one can construct a set of graphs with critical probabilities dense in $(0, 1)$.

1. Introduction

In the area of thermal critical phenomena there is evidence from many different sources that critical exponents depend on the dimension of a problem, but not on the particular lattice in that dimension. Partly because of the relationship to the Potts model (Kasteleyn and Fortuin 1969) and partly because of direct numerical evidence (Gaunt and Sykes 1983, and references therein) it is believed that exponents in percolation theory are universal, in this sense. However, unlike the two-dimensional Ising problem, it has not yet been *proved* for the percolation problem that exponents are the same for *any* pair of common lattices.

Let Γ be an infinite connected graph. In the bond percolation problem on Γ we consider bonds (i.e. edges of Γ) to be *open* with probability p or *closed* with probability $q = 1-p$ and we write the probability that a randomly chosen bond is open and connected to an infinite cluster of open bonds, in the form $pP(p)$. The *percolation probability* $P(p)$ satisfies $P(0)=0$ and $P(1)=1$ and we define the *critical probability* p_c to be

$$p_c = \sup\{p: P(p)=0\}. \tag{1.1}$$

It is often assumed that the *critical exponent*

$$\beta = \lim_{p\to p_c^+} \frac{\log P(p)}{\log (p-p_c)} \tag{1.2}$$

exists and, if it does, we write

$$P(p) \sim (p-p_c)^\beta \qquad \text{as} \qquad p\to p_c^+. \tag{1.3}$$

For the square lattice in two dimensions it is known that $p_c = \frac{1}{2}$ (Kesten 1980) and it is suspected that β exists and that $\beta \simeq 0.14$. Recently van den Berg (1982) and Grimmett (1983) have shown how to produce new graphs from the square lattice graph

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in such a way as to vary p_c. We add a new technique for varying p_c called *bond decoration* and show that β remains unchanged when this construction is applied. A preliminary account of some of these results has appeared (Ord and Whittington 1980).

2. Bond decoration

The basic idea in bond decoration is to replace each bond in an *initial graph* Γ by a finite, connected, two-rooted graph G and thereby produce a *decorated graph* $\Gamma^* = \Gamma^G$. The probability that there is a path of open bonds in G joining the roots of G is called the *transmission function* $f_G(p)$. If Γ has critical probability p_c then $\Gamma^* = \Gamma^G$ will have critical probability p_c^* where $f(p_c^*) = p_c$. The stability of β depends on the fact that the percolation probability of Γ^* can be written in the form

$$P^*(p) = h_G(p)P(f_G(p)) \tag{2.1}$$

where $h_G(p)$ is the *association function*, the probability that an open bond in G is connected by a path of open bonds to at least one root vertex.

Transmission functions and association functions play an important role in what follows so we include a specific example. If G is the rooted graph shown in figure 1 then

$$\begin{aligned} f(p) &= 2p^2q^3 + 8p^3q^2 + 5p^4q + p^5 \\ &= 2p^2(1-p)^3 + 8p^3(1-p)^2 + 5p^4(1-p) + p^5 \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} h(p) &= \tfrac{4}{5} + \tfrac{1}{5}(1-q^4) \\ &= \tfrac{4}{5} + \tfrac{1}{5}(1-(1-p)^4). \end{aligned} \tag{2.3}$$

Notice that f is strictly increasing from $f(0) = 0$ to $f(1) = 1$ and $h(p) \neq 0$.

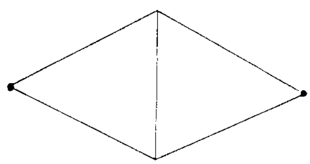

Figure 1. An example of a two rooted decorating graph.

The fact that $h(p) \neq 0$ for arbitrary graphs G is trivial since a lower bound for $h(p)$ is given by the fraction of bonds in G which are joined directly to one root vertex or the other. The crucial fact about transmission functions is more subtle and so we isolate it in the following.

Lemma 1. Let G be a finite, connected graph with root vertices A and B. Then the transmission function $f(p)$ satisfies $f'(p) > 0$ for $0 < p < 1$ and $f'(p) \geq 0$ for $p = 0$ and 1. The inequality at $p = 0$ is strict iff A and B are joined by an edge and the inequality at $p = 1$ is strict iff there is at least one edge in G whose removal disconnects A and B.

Proof. The key observation is that adding a bond to a configuration of bonds which already joins A and B does not spoil this property. If there are $0 \leq F_k \leq \binom{n}{k}$ favourable configurations with k open bonds then

$$f(p)=\sum_{k=0}^{n} F_k p^k q^{n-k} \tag{2.4}$$

and the derivative of this with respect to p is

$$f'(p)=\sum_{k=0}^{n-1} [(k+1)F_{k+1}-(n-k)F_k]p^k q^{(n-1)-k}. \tag{2.5}$$

We claim that $(k+1)F_{k+1}-(n-k)F_k \geq 0$ and hence $f'(p)\geq 0$. This is so because $(k+1)F_{k+1}$ counts the total number of favourable configurations with $k+1$ open bonds and one of these open bonds distinguished, while $(n-k)F_k$ counts the subset of these which arise by adding a distinguished open bond to a favourable configuration with k open bonds.

Since $F_0=0$ and $F_n=1$ we have

$$\begin{aligned} &f(0)=0, && f'(0)=F_1;\\ &f(1)=1, && f'(1)=n-F_{n-1}; \end{aligned} \tag{2.6}$$

and, for $0<p<1$,

$$f'(p)\geq k_0 F_{k_0} p^{k_0-1} q^{n-k_0}>0 \tag{2.7}$$

where $k_0=\min\{k\colon F_k\neq 0\}$ satisfies $1\leq k_0\leq n$. The transmission function satisfies $f'(0)>0$ iff A and B are joined by an edge (for otherwise $F_1=0$) and $f'(1)>0$ iff there is at least one edge whose removal disconnects A and B (for otherwise $F_{n-1}=n$).

Lemma 1 allows us to prove the following.

Theorem. Let Γ be an infinite, connected graph with critical probability p_c satisfying $0<p_c<1$. Let G be a finite, connected, two-rooted graph with transmission function $f(p)$ and association function $h(p)$. Then the decorated graph $\Gamma^*=\Gamma^G$ has critical probability p_c^* defined by the equation $f(p_c^*)=p_c$. Moreover if Γ has critical exponent β then so does Γ^*.

Proof. There is a natural mapping from configurations on Γ^* to configurations on Γ in which a bond is declared open on Γ iff the roots of the corresponding copy of G are joined within that copy of G. Thus a bond probability of p on Γ^* corresponds to a bond probability of $f(p)$ on Γ and percolation occurs on Γ^* at the unique solution $p=p_c^*$ of the equation $f(p)=p_c$.

We have already mentioned that Γ^* has percolation probability

$$P^*(p)=h(p)P(f(p)). \tag{2.8}$$

If Γ has critical exponent

$$\beta=\lim_{t\to p_c^+}\frac{\log P(t)}{\log(t-p_c)} \tag{2.9}$$

then we can use the fact that $h(p)\geq c>0$ and $f(p)$ is continuously differentiable and satisfies $f'(p_c^*)>0$ to prove that Γ^* also has critical exponent β. The details are as follows:

$$\lim_{p\to p_c^*+}\frac{\log P^*(p)}{\log(p-p_c^*)}$$

$$= \lim_{p\to p_c^*+}\left(\frac{\log h(p)}{\log(p-p_c^*)}+\frac{\log P(f(p))}{\log(f(p)-p_c)}\cdot\frac{\log(f(p)-p_c)}{\log(p-p_c^*)}\right)$$

$$=0+\left(\lim_{t\to p_c+}\frac{\log P(t)}{\log(t-p_c)}\right)\cdot\left(\lim_{p\to p_c^*+}\frac{\log(f(p)-f(p_c^*))}{\log(p-p_c^*)}\right)$$

$$=\beta\left(\lim_{p,p'\to p_c^*+}\frac{\log f'(p')+\log(p-p_c^*)}{\log(p-p_c^*)}\right)$$

$$=\beta\cdot[0+1]=\beta \qquad (2.10)$$

Now we may ask what values of p_c^* can be achieved by the method of the theorem. The answer is a countable, dense subset of (0, 1). The countability comes from the fact that there are only countably many finite, connected, two-rooted graphs G and hence only countably many transmission functions f_G. The density of the set of solutions of equations $f_G(p)=p_c$ for fixed p_c and variable G comes from the following explicit construction.

One may consider building up finite, connected, two-rooted graphs G by stringing together smaller graphs of this type in series or in parallel. If G_i $(i=1,2,\ldots,k)$ has transmission function f_i then the transmission function which results from a series construction is

$$f(p)=\prod_{i=1}^{k} f_i(p) \qquad (2.11)$$

and the transmission function which results from a parallel construction is

$$f(p)=1-\prod_{i=1}^{k}(1-f_i(p)). \qquad (2.12)$$

In particular, the transmission function which results from a series of n bonds is p^n and the result of putting m of these together in parallel is

$$f_{mn}(p)=1-(1-p^n)^m. \qquad (2.13)$$

Our assertion about the density of p_c^* values follows from

Lemma 2. For fixed p_c, $0<p_c<1$ and for any constants a and b, $0<a<b<1$, it is possible to choose positive integers m and n so that the solution of the equation $f_{mn}(p)=p_c$ lies in the interval $a\leqslant p\leqslant b$.

Proof. Consider the intervals (a^j, b^j) $j=1,2,3,\ldots$. These overlap for $j\geqslant k$ where $b^{k+1}>a^k$, that is, for $k>\log b/(\log a-\log b)$, and so cover the interval $(0, b^k)$.

Since $\lim_{l\to\infty}(1-p_c)^{1/l}=1$ it is possible to choose m so that $1-(1-p_c)^{1/m}<b^k$. But this means that there is an n with the property that $a^n\leqslant 1-(1-p_c)^{1/m}\leqslant b^n$. If we define p by setting $p^n=1-(1-p_c)^{1/m}$ then $f_{mn}(p)=p_c$ and $a\leqslant p\leqslant b$.

The lemma shows that finite series–parallel graphs are already enough to produce a dense subset of p_c^* values. It is significant that these are planar graphs because this means that there is a collection of planar decorations of the square lattice graph with p_c^* values dense in (0, 1). If the square lattice graph does indeed have a critical exponent $\beta \approx 0.14$ then so do all these decorated modifications.

3. Discussion

We have shown that if the critical exponent β exists, it is invariant under decoration of an infinite connected graph Γ, by a finite connected two-rooted graph G. Such decorations change the critical percolation probability and we have shown that, by varying G, one can construct a set of graphs with critical probabilities dense in (0, 1). Moreover, all these graphs have the same critical exponent, β.

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References

Gaunt D S and Sykes M F 1983 *J. Phys. A: Math. Gen.* **16** 783
Grimmett G R 1983 *J. Phys. A: Math. Gen.* **16** 599
Kasteleyn P W and Fortuin C M 1969 *J. Phys. Soc. Japan Suppl.* **26** 11
Kesten H 1980 *Commun. Math. Phys.* **74** 41
Ord G and Whittington S G 1980 *J. Phys. A: Math. Gen.* **13** L307
van den Berg J 1982 *J. Phys. A: Math. Gen.* **15** 605